Non-parametric Empirical Bayes and Compound Bayes Estimation of Independent Normal Means

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Abstract

Increasing attention on applications involving problems with many unknown parameters (large *p*) has led to a renewed interest in empirical Bayes and Compound Bayes approaches like those first proposed in Robbins (1951, 1956). In this talk we describe a new method for the prototypical version of the nonparametric Empirical Bayes estimation problem.

Consider the classical problem of estimating a vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)'$, based on independent observations $Y_i \sim N(\mu_i, \sigma^2)$. Suppose the μ_i are themselves independent realizations from a completely unknown distribution G_p. We propose an easily computed estimator $\tilde{\boldsymbol{\mu}}$ and study the ratio of its expected risk $E_{G_p} E_{\boldsymbol{\mu}} \left(\| \tilde{\boldsymbol{\mu}} - \boldsymbol{\mu} \|^2 \right)$ to that of the Bayes procedure. We show under very mild conditions that this ratio approaches 1 as $p \to \infty$. A related compound decision theoretic formulation is also studied. There, no prior is assumed and this estimator is asymptotically optimal relative to the best possible estimator given the values of the order statistics $\boldsymbol{\mu}_{(*),p} = (\mu_{(1)}, \ldots, \mu_{(p)})'$.

There has been much contemporary interest in estimators that are valid in sparse settings; settings such as those for which G_p is concentrated on two points, with heaviest probability at $\mu = 0$. Formally, this has $G_p(\{u\}) = 1 - \gamma_p$ if $u = 0$, $= \gamma_p$ if $u = u_p$ and lets $\gamma_p \to 0$. The conditions on the sequences $\{G_p\}$ or $\{\boldsymbol{\mu}_{(*),p}\}$ for asymptotic optimality of our $\tilde{\boldsymbol{\mu}}$ are only mildly restrictive, and include a broad range of problems involving sparsity. In particular, our proposed estimator is asymptotically optimal in moderately "sparse" settings – ones such as those described just above in which $p\gamma_n \to \infty$ and $p(1-\gamma_n) \to \infty$ and $0 < \liminf u_p$, $\limsup u_p < \infty$.

A simulation study is presented to demonstrate the performance of our estimator. In moderately sparse settings our estimator performs very well in comparison with current procedures tailored for sparse situations. It also adapts well to non-sparse situations.

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